Some relations between the topological and geometric filtration for smooth projective varieties

Wenchuan Hu

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Contents

Abstract

In the first part of this paper, we show that the assertion "$T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$" (which is called the Friedlander-Mazur conjecture) is a birationally invariant statement for smooth projective varieties X when $p = \dim(X) - 2$ and when $p = 1$. We also establish the Friedlander-Mazur conjecture in certain dimensions. More precisely, for a smooth projective variety X, we show that the topological filtration $T_pH_{2p+1}(X,\mathbb{Q})$ coincides with the geometric filtration $G_pH_{2p+1}(X,\mathbb{Q})$ for all p. (Friedlander and Mazur had previously shown that $T_pH_{2p}(X,\mathbb{Q}) = G_pH_{2p}(X,\mathbb{Q})$). As a corollary, we conclude that for a smooth projective threefold X, $T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$ for all $k \geq 2p \geq 0$ except for the case $p = 1, k = 4$. Finally, we show that the topological and geometric filtrations always coincide if Suslin's conjecture holds.

1 Introduction

In this paper, all varieties are defined over $\mathbb{C}$. Let X be a projective variety with dimension n. Let $\mathcal{Z}_p(X)$ be the space of algebraic p-cycles.

The **Lawson homology** $L_pH_k(X)$ of p-cycles is defined by

$$L_pH_k(X) = \pi_{k-2p}(\mathcal{Z}_p(X)) \quad for \quad k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology (cf. [F1], [L1]). For general background, the reader is referred to Lawson' survey paper [L2].

In [FM], Friedlander and Mazur showed that there are natural maps, called **cycle class maps**

$$\Phi_{p,k} : L_pH_k(X) \to H_k(X).$$

Definition 1.1

$$\begin{aligned}&L_pH_k(X)_{hom} := \ker\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\};\\ &T_pH_k(X) := \mathrm{Image}\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\};\\ &T_pH_k(X,\mathbb{Q}) := T_pH_k(X) \otimes \mathbb{Q}.\end{aligned}$$

It was shown in [[FM], §7] that the subspaces $T_pH_k(X,\mathbb{Q})$ form a decreasing filtration:

$$\cdots \subseteq T_pH_k(X,\mathbb{Q}) \subseteq T_{p-1}H_k(X,\mathbb{Q}) \subseteq \cdots \subseteq T_0H_k(X,\mathbb{Q}) = H_k(X,\mathbb{Q})$$

and $T_pH_k(X,\mathbb{Q})$ vanishes if $2p > k$.

Definition 1.2 ([FM]) *Denote by $G_pH_k(X,\mathbb{Q}) \subseteq H_k(X,\mathbb{Q})$ the $\mathbb{Q}$-vector subspace of $H_k(X,\mathbb{Q})$ generated by the images of mappings $H_k(Y,\mathbb{Q}) \to H_k(X,\mathbb{Q})$, induced from all morphisms $Y \to X$ of varieties of dimension $\leq k-p$.*

*The subspaces $G_pH_k(X,\mathbb{Q})$ also form a decreasing filtration (called **geometric filtration**):*

$$\cdots \subseteq G_pH_k(X,\mathbb{Q}) \subseteq G_{p-1}H_k(X,\mathbb{Q}) \subset \cdots \subseteq G_0H_k(X,\mathbb{Q}) \subseteq H_k(X,\mathbb{Q})$$

If X is smooth, the Weak Lefschetz Theorem implies that $G_0H_k(X,\mathbb{Q}) = H_k(X,\mathbb{Q})$. Since $H_k(Y,\mathbb{Q})$ vanishes for k greater than twice the dimension of Y, $G_pH_k(X,\mathbb{Q})$ vanishes if $2p > k$.

The following results have been proved by Friedlander and Mazur in [FM]:

Theorem 1.1 ([FM]) *Let X be any projective variety.*

1. *For non-negative integers p and k,*

$$T_pH_k(X,\mathbb{Q}) \subseteq G_pH_k(X,\mathbb{Q}).$$

2. *When $k = 2p$,*

$$T_pH_{2p}(X,\mathbb{Q}) = G_pH_{2p}(X,\mathbb{Q}).$$

Question ([FM], [L2]): Does one have equality in Theorem 1.1 when X is a smooth projective variety?

Friedlander [F2] has the following result:

Theorem 1.2 ([F2]) *Let X be a smooth projective variety of dimension n. Assume that Grothendieck's Standard Conjecture B ([Gro]) is valid for a resolution of singularities of each irreducible subvariety of $Y \subset X$ of dimension $k-p$, then*

$$T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q}).$$

Remark 1.1 ([Lew],§15.32) *The Grothendieck's Standard Conjecture B is known to hold for a smooth projective variety X in the following cases:*

1. $\dim X \leq 2$.
2. Flag manifolds X.
3. Smooth complete intersections X.
4. Abelian varieties (due to D. Lieberman [Lieb]).

In this paper, we will use the tools in Lawson homology and the methods given in [H] to show the following main results:

Theorem 1.3 *Let X be a smooth projective variety of dimension n. If the conclusion in Theorem 1.2 holds (without the assumption of Grothendieck's Standard Conjecture B) for X with $p = 1$,(resp.$p = n-2$) (k arbitrary), then it also holds for any smooth projective variety X' which is birationally equivalent to X with $p = 1$,(resp.$p = n-2$).*

Theorem 1.4 *For any smooth projective variety X,*

$$T_pH_{2p+1}(X,\mathbb{Q}) = G_pH_{2p+1}(X,\mathbb{Q}).$$

As corollaries, we have

Corollary 1.1 *Let X be a smooth projective 3-fold. We have $T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$ for all $k \geq 2p \geq 0$ except for the case $p = 1, k = 4$.*

Corollary 1.2 *Let X be a smooth projective 3-fold with $H^{2,0}(X) = 0$. Then $T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$ for any $k \geq 2p \geq 0$. In particular, it holds for X a smooth hypersurface and a complete intersection of dimension 3.*

By using the Künneth formula in homology with rational coefficient, we have

Corollary 1.3 *Let X be the product of a smooth projective curve and a smooth simply connected projective surface. Then $T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$ for any $k \geq 2p \geq 0$.*

Corollary 1.4 *For 4-folds X, the assertion that $T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$ holds for all $k \geq 2p \geq 0$ is a birational invariant statement. In particular, if X is a rational manifold with $\dim(X) \leq 4$, then the conclusion in Theorem 1.2 holds for any $k \geq 2p \geq 0$ without assumption of Grothendieck's Standard Conjecture B .*

Remark 1.2 *A Conjecture given by Suslin (see [FHW], §7) implies that $L_pH_{n+p}(X^n) \cong H_{n+p}(X^n)$.*

As an application of Theorem 1.4 and Proposition 3.1, we have the following result:

Corollary 1.5 *If the Suslin's Conjecture is true, then the topological filtration is the same as the geometric filtration for a smooth projective variety.*

The main tools to prove this result are: the long exact localization sequence given by Lima-Filho in [Li], the explicit formula for Lawson homology of codimension-one cycles on a smooth projective manifold given by Friedlander in [F1], (and its generalization to general irreducible varieties, see below), and the weak factorization theorem proved by Wlodarczyk in [W] and in [AKMW].

2 The Proof of the Theorem 1.3

Let X be a smooth projective manifold of dimension n and $i_0 : Y \hookrightarrow X$ be a smooth subvariety of codimension $r \geq 2$. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of X along Y, $\pi : D = \sigma^{-1}(Y) \to Y$ the nature map, and $i : D = \sigma^{-1}(Y) \hookrightarrow \tilde{X}_Y$ the exceptional divisor of the blowup. Set $U := X - Y \cong \tilde{X}_Y - D$. Denote by j_0 the inclusion $U \subset X$ and j the inclusion $U \subset \tilde{X}_Y$.

Now I list the Lemmas and Corollaries given in [H].

Lemma 2.1 *For each $p \geq 0$, we have the following commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \to & L_pH_k(D) & \stackrel{i_*}{\to} & L_pH_k(\tilde{X}_Y) & \stackrel{j^*}{\to} & L_pH_k(U) & \stackrel{\delta_*}{\to} & L_pH_{k-1}(D) & \to \cdots \\
& \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow\cong & & \downarrow \pi_* & \\
\cdots \to & L_pH_k(Y) & \stackrel{(i_0)_*}{\to} & L_pH_k(X) & \stackrel{j_0^*}{\to} & L_pH_k(U) & \stackrel{(\delta_0)_*}{\to} & L_pH_{k-1}(Y) & \to \cdots
\end{array}$$

Remark 2.1 *Since π_* is surjective (there is an explicitly formula for the Lawson homology of D, i.e., the Projective Bundle Theorem proved by Friedlander and Gabber, see [FG]), it is easy to see that σ_* is surjective.*

Corollary 2.1 *If $p = 0$, then we have the following commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \to & H_k(D) & \stackrel{i_*}{\to} & H_k(\tilde{X}_Y) & \stackrel{j^*}{\to} & H_k^{BM}(U) & \stackrel{\delta_*}{\to} & H_{k-1}(D) & \to \cdots \\
& \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow\cong & & \downarrow \pi_* & \\
\cdots \to & H_k(Y) & \stackrel{(i_0)_*}{\to} & H_k(X) & \stackrel{j_0^*}{\to} & H_k^{BM}(U) & \stackrel{(\delta_0)_*}{\to} & H_{k-1}(Y) & \to \cdots
\end{array}$$

Moreover, if $x \in H_k(D)$ maps to zero under π_* and i_*, then $x = 0 \in H_k(D)$.

Corollary 2.2 *If $p = n-2$, then we have the following commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \to & L_{n-2}H_k(D) & \xrightarrow{i_*} & L_{n-2}H_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_{n-2}H_k(U) & \xrightarrow{\delta_*} & L_{n-2}H_{k-1}(D) & \to \cdots \\
& \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \downarrow \pi_* & \\
\cdots \to & L_{n-2}H_k(Y) & \xrightarrow{(i_0)*} & L_{n-2}H_k(X) & \xrightarrow{j_0^*} & L_{n-2}H_k(U) & \xrightarrow{(\delta_0)_*} & L_{n-2}H_{k-1}(Y) & \to \cdots
\end{array}$$

Lemma 2.2 *For each $p \geq 0$, we have the following commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \to & L_pH_k(D) & \xrightarrow{i_*} & L_pH_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_pH_k(U) & \xrightarrow{\delta_*} & L_pH_{k-1}(D) & \to \cdots \\
& \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & \\
\cdots \to & H_k(D) & \xrightarrow{i_*} & H_k(\tilde{X}_Y) & \xrightarrow{j^*} & H_k^{BM}(U) & \xrightarrow{\delta_*} & H_{k-1}(D) & \to \cdots
\end{array}$$

In particular, it is true for $p = 1, n-2$.

Proof. See [Li] and also [FM].

□

Lemma 2.3 *For each $p \geq 0$, we have the following commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \to & L_pH_k(Y) & \xrightarrow{(i_0)_*} & L_pH_k(X) & \xrightarrow{j^*} & L_pH_k(U) & \xrightarrow{(\delta_0)_*} & L_pH_{k-1}(Y) & \to \cdots \\
& \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & \\
\cdots \to & H_k(Y) & \xrightarrow{(i_0)_*} & H_k(X) & \xrightarrow{j^*} & H_k^{BM}(U) & \xrightarrow{(\delta_0)_*} & H_{k-1}(Y) & \to \cdots
\end{array}$$

In particular, it is true for $p = 1, n-2$.

Proof. See [Li] and also [FM]. □

Remark 2.2 *The smoothness of X and Y is not necessary in the Lemma 2.3.*

Remark 2.3 *All the commutative diagrams of long exact sequences above remain commutative and exact when tensored with $\mathbb{Q}$. We will use these Lemmas and Corollaries with rational coefficients.*

The following result will be used several times in the proof of our main theorem:

Theorem 2.1 *(Friedlander* [F1]*) Let W be any smooth projective variety of dimension n. Then we have the following isomorphisms*

$$\begin{cases}
L_{n-1}H_{2n}(W) \cong \mathbf{Z}, \\
L_{n-1}H_{2n-1}(W) \cong H_{2n-1}(X, \mathbf{Z}), \\
L_{n-1}H_{2n-2}(W) \cong H_{n-1,n-1}(X, \mathbf{Z}) = NS(W) \\
L_{n-1}H_k(X) = 0 \quad for \quad k > 2n.
\end{cases}$$

The proof of Theorem 1.3 ($p = n-2$):

There are two cases:

Case 1. If $T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$, then $T_pH_k(\tilde{X}_Y,\mathbb{Q}) = G_pH_k(\tilde{X}_Y,\mathbb{Q})$.

The injectivity of $T_pH_k(\tilde{X}_Y,\mathbb{Q}) \to G_pH_k(\tilde{X}_Y,\mathbb{Q})$ has been proved by Friedlander and Mazur in [FM]. We only need to show the surjectivity. Note that the case for $k = 2p+1$ holds for any smooth projective variety (Theorem 1.4). We only need to consider the cases where $k \geq 2p+2$. In these cases, $k - p \geq p+2 = n$, from the definition of the geometric filtrations, we have $G_pH_k(\tilde{X},\mathbb{Q}) = H_k(\tilde{X}_Y,\mathbb{Q})$ and $G_pH_k(X,\mathbb{Q}) = H_k(X,\mathbb{Q})$.

Let $b \in G_pH_k(\tilde{X}_Y,\mathbb{Q})$, and a be the image of b under the the map $\sigma_* : H_k(\tilde{X}_Y,\mathbb{Q}) \to H_k(X,\mathbb{Q})$, i.e., $\sigma_*(b) = a$. By assumption, there exists an element $\tilde{a} \in L_{n-2}H_k(X) \otimes \mathbb{Q}$ such that $\Phi_{n-2,k}(\tilde{a}) = a$. Since $\sigma_* : L_{n-2}H_k(\tilde{X}_Y) \otimes \mathbb{Q} \to L_{n-2}H_k(X) \otimes \mathbb{Q}$ is surjective ([H]), there exists an element $\tilde{b} \in L_{n-2}H_k(X) \otimes \mathbb{Q}$ such that $\sigma_*(\tilde{b}) = \tilde{a}$. By the following commutative diagram

$$\begin{array}{ccc} L_{n-2}H_k(\tilde{X}_Y) \otimes \mathbb{Q} & \xrightarrow{\sigma_*} & L_{n-2}H_k(X) \otimes \mathbb{Q} \\ \downarrow \Phi_{n-2,k} & & \downarrow \Phi_{n-2,k} \\ H_k(\tilde{X}_Y,\mathbb{Q}) & \xrightarrow{\sigma_*} & H_k(X,\mathbb{Q}), \end{array}$$

we have $\Phi_{n-2,k}(\tilde{b}) - b$ maps to zero in $H_k(X,\mathbb{Q})$. By the commutative diagram in Corollary 2.1, $j^*(\Phi_{n-2,k}(\tilde{b}) - b) = 0 \in H_k^{BM}(U,\mathbb{Q})$. From the exactness of the upper long exact sequence in Corollary 2.1, there exists an element $c \in H_k(D,\mathbb{Q})$ such that $i_*(c) = \Phi_{n-2,k}(\tilde{b}) - b$. From Theorem 2.1, we find that $\Phi_{n-2,k} : L_{n-2}H_k(D) \otimes \mathbb{Q} \to H_k(D) \otimes \mathbb{Q}$ is an isomorphism for $k \geq 2n-2$. Hence there exists an element $\tilde{c} \in L_{n-2}H_k(D) \otimes \mathbb{Q}$ such that $i_*(\Phi_{n-2,k}(\tilde{c})) = \Phi_{n-2,k}(\tilde{b}) - b$. Therefore $\Phi_{n-2,k}(\tilde{b} - i_*(\tilde{c})) = b$, i.e., the surjectivity of $T_pH_k(\tilde{X}_Y,\mathbb{Q}) \to G_pH_k(\tilde{X}_Y,\mathbb{Q})$.

On the other hand, we need to show

Case 2. If $T_pH_k(\tilde{X}_Y,\mathbb{Q}) = G_pH_k(\tilde{X}_Y,\mathbb{Q})$, then $T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$.

This part is relatively easy. By Theorem 1.4, we only need to consider the cases that $k \geq 2p+2 = 2n-2$. Let $a \in G_pH_k(X,\mathbb{Q}) = H_k(X,\mathbb{Q})$. From the blow up formula for singular homology (cf. [GH]), we know $\sigma_* : H_k(\tilde{X}_Y,\mathbb{Q}) \to H_k(X,\mathbb{Q})$ is surjective. Then there exists an element $b \in H_k(\tilde{X}_Y,\mathbb{Q})$ such that $\sigma_*(b) = a$. By assumption, we can find an element $\tilde{b} \in L_{n-2}H_k(\tilde{X}_Y,\mathbb{Q})$ such that $\Phi_{n-2,k}(\tilde{b}) = b$. Set $\tilde{a} = \sigma_*(\tilde{b})$. Then $\Phi_{n-2,k}(\tilde{a}) = a$ under the natural map $\Phi_{n-2,k}$. This is exactly the surjectivity we want.

This completes the proof for a blow-up along a smooth codimension at least two subvariety Y in X.

$\square$

The proof of Theorem 1.3 ($p = 1$):

The injectivity of the map $T_1H_k(W,\mathbb{Q}) \to G_1H_k(W,\mathbb{Q})$ has been proved for any smooth projective variety W by Friedlander and Mazur in [FM]. We only need to show the surjectivity under certain assumption.

Similar to the case $p = n-2$, we also have two cases:

Case A. If $T_1H_k(X,\mathbb{Q}) = G_1H_k(X,\mathbb{Q})$, then $T_1H_k(\tilde{X}_Y,\mathbb{Q}) = G_1H_k(\tilde{X}_Y,\mathbb{Q})$.

From Theorem 1.4, the case where $k=3$ holds for any smooth projective variety. We only need to consider the cases where $k \geq 4$.

Let $b \in G_1H_k(\tilde{X}_Y,\mathbb{Q})$. Denote by a the image of b under the the map $\sigma_* : H_k(\tilde{X}_Y,\mathbf{Q}) \to H_k(X,\mathbb{Q})$, i.e., $\sigma_*(b) = a$. From the blow up formula for singular homology and the definition of the geometric filtration, we have $\sigma_*(G_1H_k(\tilde{X}_Y,\mathbb{Q})) = G_1H_k(X,\mathbb{Q})$.

By assumption, there exists an element $\tilde{a} \in L_1H_k(X)\otimes\mathbb{Q}$ such that $\Phi_{1,k}(\tilde{a}) = a$. Since $\sigma_* : L_1H_k(\tilde{X}_Y)\otimes\mathbb{Q} \to L_1H_k(X)\otimes\mathbb{Q}$ is surjective ([H]), there exists an element $\tilde{b} \in L_1H_k(\tilde{X}_Y)\otimes\mathbb{Q}$ such that $\sigma_*(\tilde{b}) = \tilde{a}$. By the following commutative diagram

$$\begin{array}{ccc} L_1H_k(\tilde{X}_Y)\otimes\mathbb{Q} & \stackrel{\sigma_*}{\to} & L_1H_k(X)\otimes\mathbb{Q} \\ \downarrow \Phi_{1,k} & & \downarrow \Phi_{1,k} \\ H_k(\tilde{X}_Y,\mathbb{Q}) & \stackrel{\sigma_*}{\to} & H_k(X,\mathbb{Q}), \end{array}$$

we have $\Phi_{1,k}(\tilde{b}) - b$ maps to zero in $H_k(X,\mathbb{Q})$. By the commutative diagram in Corollary 2.1, $j^*(\Phi_{1,k}(\tilde{b})-b) = 0 \in H_k^{BM}(U,\mathbb{Q})$. From the exactness of the upper long exact sequence in Corollary 2.1, there exists an element $c \in H_k(D,\mathbb{Q})$ such that $i_*(c) = \Phi_{1,k}(\tilde{b}) - b$. Set $d = \pi_*(c) \in H_k(Y,\mathbb{Q})$. By the commutative diagram in Corollary 2.1, d maps to zero under $(i_0)_* : H_k(Y,\mathbb{Q}) \to H_k(X,\mathbb{Q})$. Hence there exists an element $e \in H_{k+1}^{BM}(U,\mathbb{Q})$ such that whose image is d under the boundary map $(\delta_0)_*$. Let $\tilde{d} \in H_k(D,\mathbb{Q})$ be the image of e under this boundary map $\delta_* : H_{k+1}^{BM}(U,\mathbb{Q}) \to H_k(D,\mathbb{Q})$. Therefore, the image of $c - \tilde{d}$ is zero under π_* in $H_k(Y,\mathbb{Q})$ and is also zero under i_* in $H_k(\tilde{X}_Y,\mathbb{Q})$. Note that D is a bundle over Y with projective spaces as fibers. From the "projective bundle theorem" for the singular homology (cf.[GH]), we have $H_k(D,\mathbb{Q}) \cong H_k(Y,\mathbb{Q}) \oplus H_{k-2}(Y,\mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y,\mathbb{Q})$. From this, we have $c - \tilde{d} \in H_{k-2}(Y,\mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y,\mathbb{Q})$. By the revised Projective Bundle Theorem ([FG], and [H] the revised case essentially due to Complex Suspension Theorem [L1]) and Dold-Thom Theorem [DT], we have $L_1H_k(D,\mathbb{Q}) \cong L_1H_k(Y,\mathbb{Q}) \oplus L_0H_{k-2}(Y,\mathbb{Q}) \oplus \cdots \oplus L_{2-r}H_{k-2r+2}(Y,\mathbb{Q}) \cong L_1H_k(Y,\mathbb{Q}) \oplus H_{k-2}(Y,\mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y,\mathbb{Q})$, where r is the codimension of Y. Since $c - \tilde{d} \in H_{k-2}(Y,\mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y,\mathbb{Q})$ and $L_0H_{k-2}(Y,\mathbb{Q}) \oplus \cdots \oplus L_{2-r}H_{k-2r+2}(Y,\mathbb{Q}) \cong H_{k-2}(Y,\mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y,\mathbb{Q})$, there exists an element $f \in L_1H_k(D,\mathbb{Q})$ such that $\Phi_{1,k}(f) = c - \tilde{d}$. Therefore we obtain $\Phi_{1,k}(\tilde{b} - i_*(f)) = b$. This is the surjectivity we need.

Case B. If $T_1H_k(\tilde{X}_Y,\mathbb{Q}) = G_1H_k(\tilde{X}_Y,\mathbb{Q})$, then $T_1H_k(X,\mathbb{Q}) = G_1H_k(X,\mathbb{Q})$.

This part is also relatively easy. Note that $k \geq 4$. Let $a \in G_1H_k(X,\mathbb{Q}) \subseteq H_k(X,\mathbb{Q})$, then there exists an element $b \in G_1H_k(\tilde{X}_Y,\mathbb{Q})$ such that $\sigma_*(b) = a$. By assumption, we can find an element $\tilde{b} \in L_1H_k(\tilde{X}_Y,\mathbb{Q})$ such that $\Phi_{1,k}(\tilde{b}) = b$. Set $\tilde{a} = \sigma_*(\tilde{b})$. Then $\Phi_{1,k}(\tilde{a}) = a$ under the natural transformation $\Phi_{1,k}$. This is exactly the surjectivity in these cases.

This completes the proof for one blow-up along a smooth codimension at least two subvariety Y in X.

□

Now recall the weak factorization Theorem proved in [AKMW] (and also [W]) as follows:

Theorem 2.2 *([AKMW] Theorem 0.1.1, [W]) Let $\varphi\colon X \to X'$ be a birational map of smooth complete varieties over an algebraically closed field of characteristic zero, which is an isomorphism over an open set U. Then f can be factored as a sequence of birational maps*

$$X = X_0 \stackrel{\varphi_1}{\to} X_1 \stackrel{\varphi_2}{\to} \cdots \stackrel{\varphi_{n+1}}{\to} X_n = X'$$

where each X_i is a smooth complete variety, and $\varphi_{i+1} : X_i \to X_{i+1}$ is either a blowing-up or a blowing-down of a smooth subvariety disjoint from U.

□

Remark 2.4 *From the proof of the Theorem 1.3, we can draw the following conclusions:*

1. *If*
$$T_rH_k(Y, \mathbb{Q}) = G_rH_k(Y, \mathbb{Q})$$
for all k is true for algebraic r-cycles with $r \geq p$ for $\dim(Y) = n$, then
$$\text{“}T_{p-1}H_k(X, \mathbb{Q}) = G_{p-1}H_k(X, \mathbb{Q}), \quad \forall k\text{”}$$
is a birationally invariant statement for smooth projective varieties X with $\dim(X) \leq n + 2$.

2. *If*
$$T_rH_k(Y, \mathbb{Q}) = G_rH_k(Y, \mathbb{Q})$$
for all k is true for r-algebraic cycles with $r \leq p$ for $\dim(Y) = n$, then
$$\text{“}T_{p+1}H_k(X, \mathbb{Q}) = G_{p+1}H_k(X, \mathbb{Q}), \quad \forall k\text{”}$$
is a birationally invariant statement for smooth projective varieties X with $\dim(X) \leq n + 2$.

3 The Proof of the Theorem 1.4

Proposition 3.1 *For any irreducible projective variety Y of dimension n, we have*

$$\begin{cases} L_{n-1}H_{2n}(X) \cong \mathbf{Z}, \\ L_{n-1}H_{2n-1}(X) \cong H_{2n-1}(X, \mathbf{Z}), \\ L_{n-1}H_{2n-2}(X) \to H_{2n-2}(X, \mathbf{Z}) \quad is \quad injective, \\ L_{n-1}H_k(X) = 0 \quad for \quad k > 2n. \end{cases}$$

Remark 3.1 *When Y is smooth projective, Friedlander have drawn a stronger conclusion, i.e., besides those in the proposition, $L_{n-1}H_{2n-2}(Y) \cong H_{n-1,n-1}(X, \mathbf{Z}) = NS(X)$.*

Proof. Set $S = Sing(Y)$, the set of singular points. Then S is the union of proper irreducible subvarieties. Set $S = (\cup_i S_i) \bigcup S'$, where $\dim(S_i) = n-1$ and S' is the union of subvarieties with dimension $\leq n-2$. Let $V = Y - S$ be the smooth open part of Y. According to Hironaka [Hi], we can find $\tilde{Y}$ such that $\tilde{Y}$ is a smooth compactification of V. Let $D = \tilde{Y} - V$. D is a divisor on $\tilde{Y}$ with normal crossing. Denote by $i_0 : S \hookrightarrow Y$ and $i : D \hookrightarrow \tilde{Y}$ the inclusions of closed sets. Denote by $j_0 : V \hookrightarrow Y$ and $j : V \hookrightarrow \tilde{Y}$ the inclusions of open sets.

There are a few cases:

Case 1: $k \geq 2n$.

By the localization long exact sequence in Lawson homology

$$\cdots \to L_{n-1}H_k(S) \to L_{n-1}H_k(Y) \to L_{n-1}H_k(V) \to L_{n-1}H_{k-1}(S) \to \cdots,$$

we have

$$L_{n-1}H_k(Y) \cong L_{n-1}H_k(V) \quad for \quad k \geq 2n$$

since $L_{n-1}H_k(S) = 0$ for $k \geq 2n-1$.

By the localization exact sequence in homology

$$\cdots \to H_k(S) \to H_k(Y) \to H_k^{BM}(V) \to H_{k-1}(S) \to \cdots,$$

we have

$$H_k(Y) \cong H_k^{BM}(V) \quad for \quad k \geq 2n$$

since $H_k(S) = 0$ for $k \geq 2n-1$. Here $H_k^{BM}(V)$ is the Borel-Moore homology.

Similarly,

$$L_{n-1}H_k(\tilde{Y}) \cong L_{n-1}H_k(V) \quad for \quad k \geq 2n$$

and

$$H_k(\tilde{Y}) \cong H_k^{BM}(V) \quad for \quad k \geq 2n.$$

Since $\tilde{Y}$ is smooth, we have $L_{n-1}H_k(\tilde{Y}) \cong H_k(\tilde{Y})$ for $k \geq 2n$(cf. [F1]). This completes the proof for the case $k \geq 2n$.

Case 2: $k = 2n-1$.

Applying Lemma 2.3 to the pair (Y, S) for $p = n-1$, we have the commutative diagram of the long exact sequence

$$\begin{array}{ccccccccc}
0 \to & L_{n-1}H_{2n-1}(Y) & \stackrel{j_0^*}{\to} & L_{n-1}H_{2n-1}(V) & \stackrel{(\delta_0)_*}{\to} & L_{n-1}H_{2n-2}(S) & \stackrel{(i_0)_*}{\to} & L_{n-1}H_{2n-2}(Y) & \to \cdots \\
 & \downarrow \Phi_{n-1,2n-1} & & \downarrow \Phi_{n-1,2n-1} & & \downarrow \Phi_{n-1,2n-2} & & \downarrow \Phi_{n-1,2n-2} & \\
0 \to & H_{2n-1}(Y) & \stackrel{j_0^*}{\to} & H_{2n-1}^{BM}(V) & \stackrel{(\delta_0)_*}{\to} & H_{2n-2}(S) & \stackrel{(i_0)_*}{\to} & H_{2n-2}(Y) & \to \cdots
\end{array} \quad (1)$$

Similarly, applying Lemma 2.3 to the pair $(\tilde{Y}, D)$ for $p = n-1$, we have the commutative diagram of the long exact sequence

$$\begin{array}{ccccccccc} 0 \to & L_{n-1}H_{2n-1}(\tilde{Y}) & \xrightarrow{j^*} & L_{n-1}H_{2n-1}(V) & \xrightarrow{\delta_*} & L_{n-1}H_{2n-2}(D) & \xrightarrow{i_*} & L_{n-1}H_{2n-2}(\tilde{Y}) & \to \cdots \\ & \downarrow \tilde{\Phi}_{n-1,2n-1} & & \downarrow \Phi_{n-1,2n-1} & & \downarrow \Phi_{n-1,2n-2} & & \downarrow \tilde{\Phi}_{n-1,2n-2} & \\ 0 \to & H_{2n-1}(\tilde{Y}) & \xrightarrow{j^*} & H^{BM}_{2n-1}(V) & \xrightarrow{\delta_*} & H_{2n-2}(D) & \xrightarrow{i_*} & H_{2n-2}(\tilde{Y}) & \to \cdots \end{array} \qquad (2)$$

Note that $\tilde{\Phi}_{n-1,2n-2} : L_{n-1}H_{2n-2}(\tilde{Y}) \to H_{2n-2}(\tilde{Y})$ is injective, $\tilde{\Phi}_{n-1,2n-1} : L_{n-1}H_{2n-1}(\tilde{Y}) \cong H_{2n-1}(\tilde{Y})$ and $\tilde{\Phi}_{n-1,2n-2} : L_{n-1}H_{2n-2}(D) \cong H_{2n-2}(D) \cong \mathbf{Z}^m$, where m is the number of irreducible varieties of D. From (2) and the Five Lemma, we have the isomorphism

$$\Phi_{n-1,2n-1} : L_{n-1}H_{2n-1}(V) \cong H^{BM}_{2n-1}(V). \qquad (3)$$

From (1), (3) and the Five Lemma, we have the following isomorphism

$$\Phi_{n-1,2n-1} : L_{n-1}H_{2n-2}(Y) \cong H_{2n-2}(Y).$$

Case 3: $k = 2n-2$.

Now the commutative diagram (1) is rewritten in the following way:

$$\begin{array}{ccccccccc} \cdots \to & L_{n-1}H_{2n-1}(V) & \xrightarrow{(\delta_0)_*} & L_{n-1}H_{2n-2}(S) & \xrightarrow{(i_0)_*} & L_{n-1}H_{2n-2}(Y) & \xrightarrow{j_0^*} & L_{n-1}H_{2n-2}(V) & \to 0 \\ & \downarrow \Phi_{n-1,2n-1} & & \downarrow \Phi_{n-1,2n-2} & & \downarrow \Phi_{n-1,2n-2} & & \downarrow \Phi_{n-1,2n-2} & \\ \cdots \to & H^{BM}_{2n-1}(V) & \xrightarrow{(\delta_0)_*} & H_{2n-2}(S) & \xrightarrow{(i_0)_*} & H_{2n-2}(Y) & \xrightarrow{j_0^*} & H^{BM}_{2n-2}(V) & \to 0 \end{array} \qquad (4)$$

In the commutative diagram (2), we can show that the injective maps

$$j^* : H_{2n-1}(\tilde{Y}) \to H^{BM}_{2n-1}(V) \qquad (5)$$

and

$$j^* : L_{n-1}H_{2n-1}(\tilde{Y}) \to L_{n-1}H_{2n-1}(V) \qquad (6)$$

are actually isomorphisms. Hence the commutative diagram (2) reduces to the following diagram:

$$\begin{array}{ccccccc} 0 \to & L_{n-1}H_{2n-2}(D) & \to & L_{n-1}H_{2n-2}(\tilde{Y}) & \to & L_{n-1}H_{2n-2}(V) & \to 0 \\ & \downarrow \Phi_{n-1,2n-2} & & \downarrow \tilde{\Phi}_{n-1,2n-2} & & \downarrow \Phi_{n-1,2n-2} & \\ 0 \to & H_{2n-2}(D) & \to & H_{2n-2}(\tilde{Y}) & \to & H^{BM}_{2n-2}(V) & \to 0 \end{array} \qquad (7)$$

To see (5) are surjective, by the exactness of the rows in (2) we only need to show that the maps $i_* : H_{2n-2}(D) \to H_{2n-2}(\tilde{Y})$ are injective. Note that $\tilde{Y}$ is a compact Kählar manifold, and the homology class of an algebraic subvariety is nontrivial in the homology of the Kählar manifold. From these, we get the injectivity of i_*. The surjectivity of (6) follows from the same reason.

We need the following lemma.

Lemma 3.1 *The natural transformation $\Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(V) \to H^{BM}_{2n-2}(V)$ is injective.*

Proof. $a \in L_{n-1}H_{2n-2}(V)$ such that $\Phi_{n-1,2n-2}(a) = 0 \in H^{BM}_{2n-2}(V)$. Since the map $j^* : L_{n-1}H_{2n-2}(\tilde{Y}) \to L_{n-1}H_{2n-2}(V)$ is surjective, there exists an element $b \in L_{n-1}H_{2n-2}(\tilde{Y})$ such that $j^*(b) = a$. Set $\tilde{b} = \Phi_{n-1,2n-2}(b) \in H_{2n-2}(\tilde{Y})$. By the commutativity of the diagram, we have $j^*(\tilde{b}) = 0$ under the map $j^* : H_{2n-2}(\tilde{Y}) \to H^{BM}_{2n-2}(V)$. By the exactness of the bottom row in the commutative diagram (7), there exists an element $\tilde{c} \in H_{2n-2}(D)$ such that the image of $\tilde{c}$ under the map $i_* : H_{2n-2}(D) \to H_{2n-2}(\tilde{Y})$ is $\tilde{b}$. Now note that $\Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(D) \to H_{2n-2}(D)$ is an isomorphism, there exists an element $c \in L_{n-1}H_{2n-2}(D)$ such that $\Phi_{n-1,2n-2}(c) = \tilde{c}$. Hence $\Phi_{n-1,2n-2}(i_*(c) - b) = 0$. Note that $\Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(\tilde{Y}) \to H_{2n-2}(\tilde{Y})$ is injective since $\tilde{Y}$ is smooth and of dimension n (cf. [F1]). Hence we get $i_*(c) = b$, i.e., b is in the image of the map $i_* : L_{n-1}H_{2n-2}(D) \to L_{n-1}H_{2n-2}(\tilde{Y})$. Therefore $a = 0$ by the exactness of the top row of the commutative diagram (7).

□

We need to show that $\Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(Y) \to H_{2n-2}(Y)$ is injective. For $a \in L_{n-1}H_{2n-2}(Y)$ such that $\Phi_{n-1,2n-2}(a) = 0 \in H_{2n-2}(Y)$. By the commutative diagram (4) and the Lemma 3.1, the image of a under $j_0^* : L_{n-1}H_{2n-2}(Y) \to L_{n-1}H_{2n-2}(V)$ is zero. Hence there exists an element $b \in L_{n-1}H_{2n-2}(S)$ such that the image of $(i_0)_* : L_{n-1}H_{2n-2}(S) \to L_{n-1}H_{2n-2}(Y)$ is a, i.e., $(i_0)_*(b) = a$. Set $\tilde{b} = \Phi_{n-1,2n-2}(b)$. Then the image of $\tilde{b}$ under the map $(i_0)_* : H_{2n-2}(S) \to H_{2n-2}(Y)$ is zero. By exactness of the bottom row in the commutative diagram (4), there exists an element $\tilde{c}$ such that its image under the map $H^{BM}_{2n-1}(V) \to H_{2n-2}(S)$ is $\tilde{b}$. By the result in **Case 2**, $\Phi_{n-1,2n-1} : L_{n-1}H_{2n-1}(V) \to H^{BM}_{2n-1}(V)$ is an isomorphism. Hence there exists an element $c \in L_{n-1}H_{2n-1}(V)$ such that $\Phi_{n-1,2n-1}(c) = \tilde{c}$. Now since $\Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(S) \to H_{2n-2}(S)$ is an isomorphism, the image of c under the map $L_{n-1}H_{2n-1}(V) \to L_{n-1}H_{2n-2}(S)$ is exactly b. Now the exactness of the top row of the commutative diagram (4) implies the vanishing of a.

The proof of the proposition is done.

□

By using this proposition, we will give a proof of Theorem 1.4.

Proof of Theorem 1.4:

For any smooth projective variety X, the injectivity of $T_pH_{2p+1}(X,\mathbb{Q}) \to G_pH_{2p+1}(X,\mathbb{Q})$ has been proved in [[FM], §7]. We only need to show the surjectivity of $T_pH_{2p+1}(X,\mathbb{Q}) \to G_pH_{2p+1}(X,\mathbb{Q})$. For any subvariety $i: Y \subset X$, we denote by $V =: X - Y$ the complementary of Y in X. We have the following commutative diagram of the long exact sequences (Lemma 2.3, or [Li]):

$$\begin{array}{ccccccccc}
\cdots \to & L_pH_{2p+1}(Y) & \to & L_pH_{2p+1}(X) & \to & L_pH_{2p+1}(V) & \to & L_pH_{2p}(Y) & \to \cdots \\
 & \downarrow \Phi_{p,2p+1} & & \downarrow \Phi_{p,2p+1} & & \downarrow \Phi_{p,2p+1} & & \downarrow \Phi_{p,2p} & \\
\cdots \to & H_{2p+1}(Y) & \to & H_{2p+1}(X) & \to & H^{BM}_{2p+1}(V) & \to & H_{2p}(Y) & \to \cdots
\end{array}$$

Obviously, the above commutative diagram holds when tensored with $\mathbb{Q}$. In the following, we only consider the commutative diagrams with $\mathbb{Q}$-coefficient.

Now let $a \in G_pH_{2p+1}(X,\mathbb{Q})$, by definition, we can assume that a lies in the image of the map $i_*: H_{2p+1}(Y,\mathbb{Q}) \to H_{2p+1}(X,\mathbb{Q})$ for some subvariety $Y \subset X$ with dimension $\dim Y = (2p+1) - p = p+1$. Hence there exists an element $b \in H_{2p+1}(Y,\mathbb{Q})$ such that $i_*(b) = a$. By the Proposition 3.1, we know that $\Phi_{p,2p+1}: L_pH_{2p+1}(Y)\otimes\mathbb{Q} \to H_{2p+1}(Y,\mathbb{Q})$ is an isomorphism. Therefore there exists an element $\tilde{b} \in L_pH_{2p+1}(Y)\otimes\mathbb{Q}$ such that $\Phi_{p,2p+1}(\tilde{b}) = b$. Set $\tilde{a} = i_*(\tilde{b})$. Then $\tilde{a}$ maps to a under the map $L_pH_{2p+1}(X)\otimes\mathbb{Q} \to H_{2p+1}(X,\mathbb{Q})$. By the definition of the topological filtration, $a \in T_pH_{2p+1}(X,\mathbb{Q})$. This completes the proof of surjectivity of $T_pH_{2p+1}(X,\mathbb{Q}) \to G_pH_{2p+1}(X,\mathbb{Q})$.

□

Remark 3.2 *In the proof of the surjectivity of Theorem 1.4, the assumption of smoothness is not necessary, more precisely, for any irreducible projective variety X, the image of the natural transformation $\Phi_{p,2p+1}: L_pH_{2p+1}(X,\mathbb{Q}) \to H_{2p+1}(X,\mathbb{Q})$ contains $G_pH_{2p+1}(X,\mathbb{Q})$.*

Remark 3.3 *Independently, M. Warker has recently also obtained this result ([Wa], Prop. 2.5]).*

Now we prove the corollaries 1.2-1.5.

The proof of Corollary 1.1: By Theorem 1.1 and 1.4, Dold-Thom Theorem and Proposition 3.1, we only need to show the cases that $p = 1, k \geq 5$. Now the following commutative diagram ([FM], Prop.6.3)

$$\begin{array}{ccc}
L_2H_k(X)\otimes\mathbb{Q} & \xrightarrow{s} & L_1H_k(X)\otimes\mathbb{Q} \\
\downarrow \Phi_{2,k} & & \downarrow \Phi_{1,k} \\
H_k(X,\mathbb{Q}) & \cong & H_k(X,\mathbb{Q}).
\end{array}$$

shows that if $L_2H_k(X)\otimes\mathbb{Q} \to H_k(X,\mathbb{Q})$ is an surjective, then $L_1H_k(X)\otimes\mathbb{Q} \to H_k(X,\mathbb{Q})$ must be surjective. Proposition 3.1 gives the needed surjectivity for $k \geq 5$ even if X is singular variety of dimension 3.

□

The proof of Corollary 1.2: By Corollary 1.1, we only need to show that $T_1H_4(X, \mathbb{Q}) = G_1H_4(X, \mathbb{Q})$. By the assumption and Poincaré duality, $H_4(X, \mathbb{Q}) \cong H_2(X, \mathbb{Q}) \cong \mathbb{Q}$. Therefore, $G_1H_4(X, \mathbb{Q}) = H_4(X, \mathbb{Q}) \cong \mathbb{Q}$ and again by the commutative diagram

$$\begin{array}{ccc} L_2H_k(X) \otimes \mathbb{Q} & \xrightarrow{s} & L_1H_k(X) \otimes \mathbb{Q} \\ \downarrow \Phi_{2,k} & & \downarrow \Phi_{1,k} \\ H_k(X, \mathbb{Q}) & \cong & H_k(X, \mathbb{Q}), \end{array}$$

we have the surjectivity of $L_1H_4(X) \otimes \mathbb{Q} \to H_4(X, \mathbb{Q})$.

□

The proof of Corollary 1.3: Suppose $X = S \times C$, where S is a smooth projective surface and C is a smooth projective curve. We only need to consider the surjectivity of $L_1H_4(X) \otimes \mathbb{Q} \to H_4(X, \mathbb{Q})$ because of Corollary 1.1. Now the Künneth formula for the rational homology of $H_4(S \times C, \mathbb{Q})$ and Theorem 2.1 for S and C gives the surjectivity in this case.

□

The proof of Corollary 1.4: This follows directly from Theorem 1.3.

□

The proof of Corollary 1.5: By Theorem 1.4, we only need to show that $T_pH_k(X, \mathbb{Q}) = G_pH_k(X, \mathbb{Q})$ for $k \geq 2p + 2$. By the definition of geometric definition, an element $a \in G_pH_k(X, \mathbb{Q})$ comes from the linear combination of elements $b_j \in H_k(Y_j, \mathbb{Q})$ for subvarieties Y_j of $\dim Y_j \leq k - p$. From the following commutative diagram

$$\begin{array}{ccc} i_* : L_pH_k(Y) \otimes \mathbb{Q} & \to & L_pH_k(X) \otimes \mathbb{Q} \\ \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} \\ i_* : H_k(Y, \mathbb{Q}) & \to & H_k(X, \mathbb{Q}), \end{array}$$

it is enough to show that $L_pH_k(Y) \to H_k(Y)$ is surjective for any irreducible subvariety $Y \subset X$ with $\dim(Y) = k - p$. By Suslin's conjecture, this is true for any smooth variety Y since $\dim(Y) = k - p$. Now we need to show that it is also true for singular irreducible varieties if the Sulin Conjecture is true.

Using induction, we will show the following lemma:

Lemma 3.2 *If the Suslin Conjecture is true for every smooth projective variety, then it is also true for every quasi-projective variety.*

Proof. Suppose that Y is an irreducible quasi-projective variety with $\dim(Y) = m$, S is an irreducible quasi-projective variety with $\dim(S) = n < m$ and

$$\begin{cases} L_pH_{n+p-1}(S) \to H_{n+p-1}(S) \quad is \quad injective, \\ L_pH_{n+q}(S) \cong H_{n+q}(S) \quad for \quad q \geq p. \end{cases}$$

Denote by $\overline{Y}$ a projective closure of Y and $S = sing(\overline{Y})$ the singular point set of $\overline{Y}$. Let $U = \overline{Y} - S$ Let $\sigma : \widetilde{Y} \to \overline{Y}$ be a desingularization of $\overline{Y}$ and denote by $D := \widetilde{Y} - U$.

The existence of a smooth $\widetilde{Y}$ is guaranteed by Hironaka [Hi]. Then D is the union of irreducible varieties with dimension $\leq m-1$.

By Lemma 2.3, we have the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & L_pH_k(Z) & \to & L_pH_k(V) & \to & L_pH_k(U) & \to & L_pH_{k-1}(Z) & \to \cdots \\
& \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & \\
\cdots \to & H_k(Z) & \to & H_k(V) & \to & H_k^{BM}(U) & \to & L_pH_{k-1}(Z) & \to \cdots,
\end{array}$$

where $U \subset V$ are quasi-projective varieties of $\dim(V) = \dim(U) = m$ and $Z = V - U$ is a closed subvariety of V.

Claim: By inductive assumption, the above commutative diagram and the Five Lemma, we have the equivalence between

$$\begin{cases} L_pH_{m+p-1}(U) \to H_{m+p-1}(U) \quad is \quad injective, \\ L_pH_{m+q}(U) \cong H_{m+q}(U) \quad for \quad q \geq p. \end{cases}$$

and

$$\begin{cases} L_pH_{m+p-1}(V) \to H_{m+p-1}(V) \quad is \quad injective, \\ L_pH_{m+q}(V) \cong H_{m+q}(V) \quad for \quad q \geq p. \end{cases}$$

The proof of the claim is obvious.

By using the claim for finite times beginning from $V = \widetilde{Y}$, we have the result for any quasi-projective variety U. The proof of Lemma 3.2 is done.

□

By Lemma 3.2, we know that the Suslin's Conjecture is also true for singular varieties. This completes the proof of Corollary 1.4.

□

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Department of Mathematics,
Stony Brook University, SUNY,
Stony Brook, NY 11794-3651
Email:wenchuan@math.sunysb.edu